

THE FINITE REPRESENTATION PROPERTY FAILS FOR COMPOSITION AND INTERSECTION

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ABSTRACT. The title theorem is proved by example: an algebra of binary relations, closed under intersection and composition, that is not isomorphic to any such algebra on a finite set.

Let K be a class of algebras for which there is a notion of “representability over a set U ”. That is, for every set U , some algebras of K are said to be **representable over U** , while others are not. We say that K has the **finite representation property** if every finite algebra in K that has a representation over *some* set has a representation over a *finite* set.

K may be defined abstractly, as a class of algebras of some particular similarity type, satisfying some conditions which, if they are all universally quantified equations, means that K is a variety. In this case some definition of representability is still required. However, if K is taken to be a class of algebras described in some concrete set-theoretical manner, then we may wish representability to simply be membership in K . An example of this type, one that fails to have the finite representation property, is considered here.¹

Let K be the class of algebras of the form $(A, ;, \cdot)$, where $;$ and \cdot are binary operations on A , such that, for some set U , A is a set of binary relations on U , and for all $a, b \in A$, $a ; b$ is the compositum of the relations a and b , in that order, while $a \cdot b$ is the intersection of a and b (in either order). In more detail, for all $a, b \in A$ we have

$$\begin{aligned} a ; b &= \{(x, y) : \text{for some } z \in U, (x, z) \in a \text{ and } (z, y) \in b\}, \\ a \cdot b &= \{(x, y) : (x, y) \in a \text{ and } (x, y) \in b\}. \end{aligned}$$

An algebra in K can be described simply as a set of relations (on some base set U) that is closed under composition and intersection. Every algebra in K is representable over some set, namely, the base set U used to specify the algebra, which may be necessarily infinite.

Theorem 1. *K does not have the finite representation property.*

We will show this by giving an example of an algebra \mathcal{A} in K that is not isomorphic to any algebra in K with a finite base set. The example is called the **point algebra** (by analogy with the relation algebra having the same name). The base set of \mathcal{A} is the set \mathbb{Q} of rational numbers, and the elements of \mathcal{A} are these three

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relations:

$$r := \{(x, y) : x, y \in \mathbb{Q} \wedge x < y\}$$

$$z := \emptyset$$

$$e := \{(x, x) : x \in \mathbb{Q}\}$$

The tables for the two operations are given below, with the entries that are actually used later enclosed in boxes:

;	z	e	r
z	z	z	z
e	z	e	r
r	z	r	r

\cdot	z	e	r
z	z	z	z
e	z	e	z
r	z	z	r

The structure of \mathcal{A} is completely specified by the two tables, and the second table is determined entirely by either of its boxed entries. That \mathcal{A} belongs to K follows from the fact that if the elements r, e, z are defined as the binary relations given above, then the two tables can be deduced from the definitions. What we do next is assume that \mathcal{A} has a representation over some set U , and show that U must be infinite.

Theorem 2. *If U is a non-empty set with distinct relations $z, r, e \subseteq U \times U$ satisfying $r; e = r = e; r$, $r; r = r$, $z; r = z = r; z$, and $r \cdot e = z$, then U is infinite.*

Proof. First we show the intersection of the identity relation on U with r is included in z , that is,

$$(1) \quad Id_U \cdot r \subseteq z.$$

To show this, we assume

$$(2) \quad (x, x) \in r$$

and derive $(x, x) \in z$. From (2) and $r = r; e$ we get $(x, x) \in r; e$, hence we know there is some $y \in U$ such that

$$(3) \quad (x, y) \in r,$$

$$(4) \quad (y, x) \in e.$$

From (4) and (2) we get $(y, x) \in e; r$, but $e; r = r$, so

$$(5) \quad (y, x) \in r.$$

Then (5) and (4) give us $(y, x) \in r \cdot e$, but $r \cdot e = z$, so

$$(6) \quad (y, x) \in z.$$

From (3) and (6) we have $(x, x) \in r; z$, but $r; z = z$, so $(x, x) \in z$. This completes the proof of (1).

Note that (1) is equivalent to $r \cdot \bar{z} \subseteq \overline{Id_U}$, i.e., the intersection of r with the complement of z (with respect to $U \times U$) is a diversity relation (included in the complement of the identity relation on U). Note also that $z \subseteq r$ and $z \subseteq e$ because $r \cdot e = z$. All three relations z, e, r must be distinct, for otherwise we do not have a representation, hence $r \cdot \bar{z} \neq \emptyset \neq e \cdot \bar{z}$. Since $r \cdot \bar{z}$ is a non-empty diversity relation, there are distinct $x_0, y \in U$ such that

$$(7) \quad (x_0, y) \in r,$$

$$(8) \quad (x_0, y) \in \bar{z}.$$

From (7) and $r = r ; r$ we know there is some $x_1 \in U$ such that

$$(9) \quad (x_0, x_1) \in r,$$

$$(10) \quad (x_1, y) \in r.$$

If $(x_0, x_1) \in z$ then $(x_0, y) \in z ; r$ by (10), but $z ; r = z$, so we get $(x_0, y) \in z$, contradicting (8). Therefore $(x_0, x_1) \in \bar{z}$, hence $(x_0, x_1) \in r \cdot \bar{z}$ by (7). Similarly, if $(x_1, y) \in z$ then $(x_0, y) \in r ; z$ by (9), but $r ; z = z$, so we get $(x_0, y) \in z$, contradicting (8). Therefore $(x_1, y) \in \bar{z}$.

So far we have in fact proved that $r \cdot \bar{z}$ is a non-empty dense diversity relation: there are distinct $x_0, x_1, y \in U$ such that $(x_0, y), (x_0, x_1), (x_1, y) \in r \cdot \bar{z}$. We have also achieved the first stage (with $n = 1$) in the construction of $y, x_0, x_1, x_2, \dots, x_n$ such that

$$(11) \quad (x_i, x_j) \in r \cdot \bar{z} \quad \text{whenever } 0 \leq i < j \leq n,$$

$$(12) \quad (x_i, y) \in r \cdot \bar{z} \quad \text{whenever } 0 \leq i \leq n.$$

We continue this construction through one more stage. Apply the density of $r \cdot \bar{z}$ to the assumption $(x_n, y) \in r \cdot \bar{z}$, obtaining some x_{n+1} such that

$$(13) \quad (x_n, x_{n+1}) \in r \cdot \bar{z},$$

$$(14) \quad (x_{n+1}, y) \in r \cdot \bar{z}.$$

Obviously (14) implies that (12) holds with $n + 1$ in place of n . To see the same for (11), let $0 \leq i < j \leq n + 1$. If $j < n + 1$ we are done, by (11), so we may assume $j = n + 1$. We wish to show $(x_i, x_{n+1}) \in r \cdot \bar{z}$. This holds by (13) if $i = n$, so assume $i < n$. We have $(x_i, x_n) \in r$ by (11) and $(x_n, x_{n+1}) \in r$ by (13), so $(x_i, x_{n+1}) \in r ; r = r$. If $(x_i, x_{n+1}) \in z$ then $(x_i, y) \in z ; r$ by (14), but $z ; r = z$, so $(x_i, y) \in z$, contradicting (12), hence $(x_i, x_{n+1}) \in \bar{z}$. Thus we have $(x_i, x_{n+1}) \in r \cdot \bar{z}$. This construction may be continued indefinitely, so U must be infinite. \square

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